

NOTES ON THE ROOTS OF EHRHART POLYNOMIALS

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ABSTRACT. We determine lattice polytopes of smallest volume with a given number of interior lattice points. We show that the Ehrhart polynomials of those with one interior lattice point have largest roots with norm of order n^2 , where n is the dimension. This improves on the previously best known bound n and complements a recent result of Braun [8] where it is shown that the norm of a root of a Ehrhart polynomial is at most of order n^2 .

For the class of 0-symmetric lattice polytopes we present a conjecture on the smallest volume for a given number of interior lattice points and prove the conjecture for crosspolytopes.

We further give a characterisation of the roots of the Ehrhart polynomials in the 3-dimensional case and we classify for $n \leq 4$ all lattice polytopes whose roots of their Ehrhart polynomials have all real part $-1/2$. These polytopes belong to the class of reflexive polytopes.

1. INTRODUCTION

Let \mathcal{P}^n be the set of all convex lattice n -polytopes in the n -dimensional Euclidean space \mathbb{R}^n with respect to the standard lattice \mathbb{Z}^n , i.e., all vertices of $P \in \mathcal{P}^n$ have integral coordinates and $\dim(P) = n$. The lattice point enumerator of a set $S \subset \mathbb{R}^n$ is denoted by $G(S)$, i.e., $G(S) = \#(S \cap \mathbb{Z}^n)$.

In 1962 Ehrhart [13] showed that for $k \in \mathbb{N}$ the lattice point enumerator $G(kP)$, $P \in \mathcal{P}^n$, is a polynomial of degree n in k where the coefficients $G_i(P)$, $0 \leq i \leq n$, depend only on P :

$$(1.1) \quad G(kP) = \sum_{i=0}^n G_i(P) k^i.$$

Moreover in [15] he proved his famous “reciprocity law”

$$(1.2) \quad G(\text{int}(kP)) = (-1)^n \sum_{i=0}^n G_i(P) (-k)^i,$$

where $\text{int}()$ denotes the interior. Two of the $n+1$ coefficients $G_i(P)$ are obvious, namely, $G_0(P) = 1$ and $G_n(P) = \text{vol}(P)$, where $\text{vol}()$ denotes the volume, i.e., the n -dimensional Lebesgue measure on \mathbb{R}^n . Also the second leading coefficient admits a simple geometric interpretation as normalized surface area of P which we present in detail in (4.1). All other coefficients $G_i(P)$, $1 \leq i \leq n-2$, have no such direct geometric meaning, except for special classes of polytopes (cf., e.g., [3, 6, 12, 19, 25, 26, 27, 28, 32]).

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A sometimes more convenient representation of $G(k, P)$ is given by a change from the monomial basis $\{x^i : i = 0, \dots, n\}$ to the basis $\{\binom{x+n-i}{n} : i = 0, \dots, n\}$:

$$(1.3) \quad G(k, P) = \sum_{i=0}^n a_i(P) \binom{k+n-i}{n}.$$

In view of (1.1) and (1.2) we get

$$(1.4) \quad \begin{aligned} a_0(P) &= 1, \quad a_1(P) = G(P) - (n+1), \quad a_n(P) = G(\text{int}(P)), \\ a_0(P) + a_1(P) + \dots + a_n(P) &= n! \text{vol}(P), \end{aligned}$$

and all $a_i(P)$ are integers. Due to Stanley's famous non-negativity theorem [35] they are also non-negative, in contrast to the $G_i(P)$'s which might be negative.

In recent years the Ehrhart polynomial was not only regarded as a polynomial for integers k , but as a formal polynomial of a complex variable $s \in \mathbb{C}$ (cf. [2, 34, 39]). Therefore, for $P \in \mathcal{P}^n$ and $s \in \mathbb{C}$ we set

$$G(s, P) = \sum_{i=0}^n G_i(P) s^i = \prod_{i=1}^n \left(1 + \frac{s}{\gamma_i(P)}\right),$$

where $-\gamma_i(P) \in \mathbb{C}$, $1 \leq i \leq n$, are the roots of the Ehrhart polynomial $G(s, P)$. In particular, for their geometric and arithmetic mean we have

$$(1.5) \quad \left(\prod_{i=1}^n \gamma_i(P)\right)^{1/n} = (1/\text{vol}(P))^{1/n}, \quad \frac{1}{n} \sum_{i=1}^n \gamma_i(P) = \frac{1}{n} \frac{G_{n-1}(P)}{\text{vol}(P)}.$$

Here we are interested in geometric interpretations of the roots and in their size. Since the volume of lattice polytopes without interior lattice points might be arbitrary large for $n \geq 3$ the norm of the roots $|\gamma_i(P)|$ might be arbitrary small (cf. (1.5)). On the other hand, we know that the volume of a lattice polytope with $l \geq 1$ interior lattice points is bounded (cf. [18, 24, 31, 40]) by a constant depending only on l and n . Thus, up to unimodular transformations, there are only finitely many different of those lattice polytopes and in this case $|\gamma_i(P)|$ can not be too small. Therefore, it seems to be reasonable to distinguish lattice polytopes with or without interior lattice points, and we define

Definition 1.1. For $l \in \mathbb{N}$ let $\mathcal{P}^n(l)$ be the set of lattice polytopes $P \in \mathcal{P}^n$ having exactly l interior lattice points, i.e., $G(\text{int}(P)) = l$. Moreover, the set of all 0-symmetric lattice polytopes $P \in \mathcal{P}^n$ is denoted by \mathcal{P}_o^n .

As already mentioned above, it was shown by Pikhurko [31] that for $P \in \mathcal{P}^n(l)$, $l \geq 1$,

$$\text{vol}(P) \leq c_n l,$$

for a constant c_n depending only on n . Hence we get

$$\left(\prod_{i=1}^n \gamma_i(P)\right)^{1/n} \geq (c_n)^{-1/n} l^{-1/n}.$$

Candidates of lattice polytopes $P \in \mathcal{P}^n(l)$, $l \geq 1$, of maximal volume are certain simplices $T(n, l) \in \mathcal{P}^n(l)$, introduced by Perles, Wills and Zaks [40] with $\text{vol}(T(n, l)) \geq (l + 1)/n! 2^{2^{n-1}}$.

In order to present a lower bound on the volume in terms of the number of interior lattice points we define for $l \in \mathbb{N}$ the simplices

$$S_n(l) = \text{conv}\left\{e_1, \dots, e_n, -l \sum_{i=1}^n e_i\right\},$$

where e_i denotes the i -th unit vector. Observe that $G(\text{int}(S_n(l))) = l$ and $\text{vol}(S_n(l)) = (nl + 1)/n!$.

Theorem 1.2. *Let $P \in \mathcal{P}^n$. Then*

$$(1.6) \quad \text{vol}(P) \geq \frac{n G(\text{int } P) + 1}{n!},$$

and the bound is best possible for any number of interior lattice points. For $G(\text{int } P) = 1$ equality holds if and only if P is unimodular isomorphic to the simplex $S_n(1)$.

The theorem above implies that for $P \in \mathcal{P}^n(l)$ the geometric mean of the roots is bounded from above by

$$\left(\prod_{i=1}^n \gamma_i(P)\right)^{1/n} \leq (n!)^{1/n} (nl + 1)^{-1/n}.$$

In the 2-dimensional case Theorem 1.2 is a direct consequence of Pick's identity $G(P) = \text{vol}(P) + \frac{1}{2}G(\text{bd } P) + 1$, where $\text{bd } P$ denotes the boundary of P [30]. In particular, equality is attained in (1.6) iff P is a lattice triangle whose vertices are the only lattice points contained in the boundary. This also shows that for $G(\text{int } P) > 1$ the extremal cases in (1.6) are not necessarily unimodular equivalent. We remark, however, that all extremal cases have the same Ehrhart polynomial.

Proposition 1.3. *Let $P \in \mathcal{P}^n(l)$, $l \geq 1$, with $\text{vol}(P) = (nl + 1)/n!$. Then $a_i(P) = a_i(S_n(l)) = l$, $1 \leq i \leq n$.*

For 0-symmetric lattice polytopes $P \in \mathcal{P}_o^n$ there is a classical upper bound on the volume due to Blichfeldt and van der Corput (cf. [16, p. 51])

$$\text{vol}(P) \leq 2^{n-1} (G(\text{int } P) + 1).$$

Lattice boxes

$$Q_n(2l - 1) = \{x \in \mathbb{R}^n : |x_1| \leq l, |x_i| \leq 1, 2 \leq i \leq n\} \in \mathcal{P}_o^n, \quad l \in \mathbb{N},$$

show that the bound is tight. As an analogue to Theorem 1.2 in the 0-symmetric case we conjecture

Conjecture 1.1. *Let $P \in \mathcal{P}_o^n$. Then*

$$\text{vol}(P) \geq \frac{2^{n-1}}{n!} (G(\text{int } P) + 1).$$

Again for $n = 2$ the inequality follows immediately from Pick's identity and the inequality is tight for any parallelogram whose vertices are the only lattice points on the boundary. It seems to be quite likely that certain crosspolytopes, i.e., $P \in \mathcal{P}_o^n$ with $2n$ vertices, are the extremal cases for the inequality above; for the family of 0-symmetric crosspolytopes we can prove the conjecture.

Proposition 1.4. *For $P \in \mathcal{P}_o^n$ with $2n$ vertices Conjecture 1.1 is true.*

One way to prove that proposition is based on the following lemma which might be of some interest in its own.

Lemma 1.5. *Let $P \in \mathcal{P}_o^n$ with $2n$ vertices. Then*

$$a_i(P) + a_{n-i}(P) \geq \binom{n}{i} (a_n(P) + a_0(P)), \quad i = 0, \dots, n.$$

Observe that on account of (1.4) Lemma 1.5 implies Proposition 1.4. As a side effect of the proof of that lemma we get

Remark 1.6. *Let $P \in \mathcal{P}_o^n$. Then $a_i(P) \geq \binom{n}{i}$ for $0 \leq i \leq n$.*

The lower bounds on $a_i(P)$ in the remark above also follow from a much deeper and much more general result of Stanley [36] on the h -vector of “symmetric” Cohen-Macaulay simplicial complexes in conjunction with a result of Betke and McMullen [7] relating the coefficients $a_i(P)$ with the h -vector of a triangulation of the polytope. Here we give a quite elementary proof which follows the method presented by Beck and Sottile in [5].

The regular unit crosspolytope $C_n^* = \text{conv}\{\pm e_i : 1 \leq i \leq n\}$ plays a special role in the context of the roots of Ehrhart polynomials. To our knowledge it was firstly shown by Kirschenhofer, Pethoe, and Tichy [21] that the real part of $-\gamma_i(C_n^*)$ is equal to $-1/2$, $1 \leq i \leq n$. This was independently proven by Bump et al. in [9, Theorem 4] and follows also from a more general result of Rodriguez-Villegas [34]. In [4, Open problem 2.42] the authors ask for other classes of lattice polytopes such that all roots of their Ehrhart polynomials have real part $-1/2$. Since C_n^* has minimal volume among all 0-symmetric lattice polytopes an obvious candidate is the simplex $S_n(1)$ in the non-symmetric case (cf. Theorem 1.2).

Theorem 1.7. *All roots of the polynomial $G(s, S_n(1))$ have real part $-1/2$. If α_n is a root of $G(s, S_n(1))$ with maximal norm, then*

$$\left| \alpha_n + \frac{1}{2} \right| = \frac{n(n+2)}{2\pi} + o(n),$$

as n tends to infinity.

In a recent paper Braun [8] proved that the roots of an Ehrhart polynomial lie inside the disc with center $-1/2$ and radius $n(n-1)/2$. The above theorem shows that this bound is essentially tight and improves on the former best known bound of order n [2, Theorem 1.3].

It seems to be quite likely that $G(s, S_n(1))$ possesses the roots of maximal norm among all Ehrhart polynomials of polytopes with interior points. In the

case $n = 2$ this follows from [2, Theorem 2.2] and for a verification of this statement in the 3-dimensional case see Theorem 1.10.

Looking at geometric properties of lattice polytopes P whose roots have all real part $-1/2$ leads immediately to the class of reflexive lattice polytopes. Here $P \in \mathcal{P}^n$ with $0 \in \text{int}P$ is called reflexive if

$$P^* = \{y \in \mathbb{R}^n : xy \leq 1, \text{ for all } x \in P\} \in \mathcal{P}^n,$$

i.e., the polar polytope is again a lattice polytope. They play an important role in toric geometry since they are in one-to-one correspondence with Gorenstein toric Fano varieties. Reflexive polytopes have been extensively studied and exhibit many surprising properties (cf. [1, 19, 29] and the references within). In particular, Hibi [19] showed that the coefficients $a_i(P)$ of a lattice polytope are symmetric, i.e., $a_i(P) = a_{n-i}(P)$, if and only if P is reflexive. Kreuzer and Skarke [22, 23] classified all reflexive polytopes in dimensions ≤ 4 . For $n = 2, 3, 4$ there are respectively 16; 4, 319 and 473, 800, 776 reflexive polytopes (up to unimodular equivalence).

Proposition 1.8. *Let $P \in \mathcal{P}^n$. If all roots of $G(s, P)$ have real part $-1/2$ then, up to an unimodular translation, P is a reflexive polytope of volume $\leq 2^n$.*

It is easy to check that for $n \leq 3$ the converse is also true but not for $n \geq 4$. All in all, for $n \leq 4$ we have the following characterization

Proposition 1.9. *Let $P \in \mathcal{P}^n$ be a reflexive polytope. Then all roots of $G(s, P)$ have real part $-1/2$*

- i) *iff $\text{vol}(P) \leq 2^n$ and $n \leq 3$,*
- ii) *iff $(G(P) - 1 - 4 \text{vol}(P))^2 \geq 16 \text{vol}(P)$, $G(P) \leq 9 \text{vol}(P) + 18$ and $n = 4$.*

A classification of the roots of 2-dimensional lattice polygons is given in the papers [2, Theorem 2.2] and [17, Theorem 1.9]. For $n = 3$ we know less and the basic properties are subsumed in the next theorem. For more detailed properties of Ehrhart polynomials of 3-dimensional lattice polytopes we refer to section 4.

Theorem 1.10. *The roots of the Ehrhart polynomials of 3-dimensional lattice polytopes are contained in*

$$[-3, -1] \cup \{a + ib : -1 \leq a < 1, a^2 + b^2 \leq 3\}$$

and the bounds on a and $a^2 + b^2$ are tight. For $P \in \mathcal{P}_3(l)$, $l \geq 1$, the upper bound $\sqrt{3}$ on the norm of the complex roots is only attained by the roots of the Ehrhart polynomial of the simplex $S_3(1)$.

The paper is organized as follows. In Section 2 we prove Theorem 1.2, Theorem 1.7 and what we know in the 0-symmetric case regarding Conjecture 1.1. Section 3 deals with reflexive polytopes and Ehrhart polynomials whose roots have all real part $-1/2$. Section 3 studies the Ehrhart polynomials and their roots for 3-dimensional lattice polytopes.

2. VOLUME AND INTERIOR LATTICE POINTS

The proof of Theorem 1.2 is based on a subdivision of P with respect to the interior lattice points contained in P .

Proof of Theorem 1.2. Let $l = G(\text{int}(P))$. If $l = 0$ there is nothing to show since any lattice polytope has at least volume $1/n!$. So let $l > 0$ and let y_1, \dots, y_l be the interior lattice points of P . Obviously, it suffices to show that P can be subdivided with the points y_1, \dots, y_k into at least $n k + 1$ lattice polytopes for $k = 1, \dots, l$.

First we build the convex hulls of y_1 with all facets of P yielding at least $n + 1$ lattice polytopes. So let us assume that we have already dissected P into $P_1, \dots, P_{n k + 1}$ lattice polytopes and let y_{k+1} be contained in the relative interior of a j -dimensional face of P_1 , say. Since y_{k+1} is an interior point it is also contained in the relative interior of a j -face of at least $n - j$ further polytopes P_2, \dots, P_r , say, $r \geq n - j + 1$. Subdividing each P_s by building the convex hull of y_{k+1} with all facets of P_s not containing y_{k+1} gives at least $j + 1$ new polytopes for each P_s , $s = 1, \dots, r$. Thus this new subdivision of P consists of at least

$$r \cdot (j + 1) + n k + 1 - r \geq (n - j + 1)j + n k + 1$$

lattice polytopes. Since $j \geq 1$ this number is at least $n(k + 1) + 1$.

The simplices $S_n(l)$ show that the bound is attained for any number of interior lattice points and the proof above shows that equality can only be achieved by simplices. So let us assume that we have a lattice simplex S with only one interior lattice point y_1 and equality in equation (1.6). Without loss of generality let $y_1 = 0$ and let v_1, \dots, v_{n+1} be the vertices of S . Let F_i be the facet of S not containing v_i , $1 \leq i \leq n + 1$. Subdividing S into the $n + 1$ simplices $\text{conv}\{0, F_i\}$, $1 \leq i \leq n + 1$, gives

$$\frac{n + 1}{n!} = \text{vol}(S) = \sum_{i=1}^{n+1} \text{vol}(\text{conv}\{0, F_i\}).$$

Since $\text{vol}(\text{conv}\{0, F_i\}) \geq 1/n!$ we must have $\text{vol}(\text{conv}\{0, F_i\}) = 1/n!$, or equivalently, any choice of n vectors out of the vertices form a basis of the lattice \mathbb{Z}^n . Thus, up to an unimodular linear transformation, we may assume $v_i = e_i$, $1 \leq i \leq n$, and the absolute value of each coordinate of v_{n+1} is 1. Finally, since 0 is contained in the interior of S we must have $v_{n+1} = (-1, \dots, -1)^T$. \square

We remark that inequality (1.6) can also be deduced from a result of Hibi [20] where it is shown that

$$(2.1) \quad a_i(P) \geq a_1(P), \quad 1 \leq i \leq n - 1,$$

for $P \in \mathcal{P}^n(l)$ with $l \geq 1$. Together with (1.4) this implies (1.6).

From (2.1) we also get Proposition 1.3, i.e., the uniqueness of the Ehrhart polynomials of $P \in \mathcal{P}^n(l)$, $l \geq 1$, with minimal volume. By (2.1) we know $a_i(P) \geq l$ for $1 \leq i \leq n$ (cf. (1.4)) and since P has minimal volume we also have

$$n! \text{vol}(S_n(l)) = 1 + n l = n! \text{vol}(P) = a_0(P) + a_1(P) + \dots + a_n(P).$$

Hence $a_i(P) = a_i(S_n(l)) = l$, $1 \leq i \leq n$, and the Ehrhart polynomial of $P \in \mathcal{P}_n(l)$ with minimal volume is uniquely determined.

We believe that the crosspolytopes

$$C_n^*(2l-1) = \text{conv}\{\pm l e_1, \pm e_2, \dots, \pm e_n\}, \quad l \geq 1,$$

with $2l-1$ interior lattice points form the 0-symmetric counterpart to the simplices $S_n(l)$, i.e., they have minimal volume among all 0-symmetric polytopes with $2l-1$ interior lattice points. In [4, Theorem 2.6] it is shown that the coefficients $a_i(P)$ of a bipyramid $P = \text{conv}\{Q, \pm e_n\}$, where Q is an $(n-1)$ -dimensional lattice polytopes embedded in the hyperplane $\{x \in \mathbb{R}^{n-1} : x_n = 0\}$ satisfy the recursion $a_i(P) = a_i(Q) + a_{i-1}(Q)$. Hence we conclude

$$a_i(C_n^*(2l-1)) = \binom{n}{i} + \binom{n-1}{i-1}(2l-2),$$

and so

$$a_i(C_n^*(2l-1)) + a_{n-i}(C_n^*(2l-1)) = \binom{n}{i} (a_n(C_n^*(2l-1)) + 1), \quad 0 \leq i \leq n.$$

Lemma 1.5 shows that $\binom{n}{i} (a_n(P) + 1)$ is a lower bound on $a_i(P) + a_{n-i}(P)$ for any lattice crosspolytope P .

Proof of Lemma 1.5. Let $P = \text{conv}\{\pm v_j : 1 \leq j \leq n\}$ be a lattice crosspolytope in \mathbb{R}^n . For any of the 2^n subsets $W_k = \{w_1, \dots, w_n\}$ with $w_j \in \{v_j, -v_j\}$ we consider the simplicial cone $C_k = \text{cone}\{(w_1, 1)^\top, \dots, (w_n, 1)^\top, e_{n+1}\} \subset \mathbb{R}^{n+1}$ and the open parallelepiped

$$Q_k = \left\{ \sum_{j=1}^n \lambda_j \begin{pmatrix} w_j \\ 1 \end{pmatrix} + \lambda_{n+1} e_{n+1} : 0 < \lambda_j < 1, 1 \leq j \leq n+1 \right\}.$$

The cones C_k form a triangulation of the cone $C = \text{cone}\{(\pm v_j, 1)^\top : 1 \leq j \leq n\}$. In a recent paper Beck and Sottile[5] introduced a new method for "calculating" the numbers $a_i(\cdot)$ of an arbitrary lattice polytope. In order to apply their approach we choose a vector $s = (s_1, \dots, s_{n+1})^\top \in \mathbb{R}^{n+1}$ such that $C \cap \mathbb{Z}^{n+1} = (s + C) \cap \mathbb{Z}^{n+1}$ and none of the shifted cones $s + C_k$ contains a lattice point on its boundary. Obviously, we must have $s_{n+1} < 0$ and the vector s can be chosen arbitrarily short. For $i = 0, \dots, n$ we denote by

$$\alpha_i(s + Q_k) = \# \{(s + Q_k) \cap \{z \in \mathbb{Z}^{n+1} : z_{n+1} = i\}\}$$

the number of lattice points in $s + Q_k$ having last coordinate i . Then we have (see [5, Proof of Theorem 2], [4, Proof of Theorem 3.12])

$$(2.2) \quad a_i(P) = \sum_k \alpha_i(s + Q_k).$$

In particular we have $a_n(P)$ many lattice points with last coordinate n contained in the parallelepipeds $s + Q_k$. Let $(w, n)^\top$ be one of them and let it be given by

$$(2.3) \quad \begin{pmatrix} w \\ n \end{pmatrix} = s + \sum_{j=1}^n \lambda_j \begin{pmatrix} w_j \\ 1 \end{pmatrix} + \lambda_{n+1} e_{n+1}$$

Now we fix an $i \in \{1, \dots, n-1\}$. For a subset $I \subset \{1, \dots, n\}$ of cardinality i we denote by I^c its complement and let

$$\lambda_I := \lambda_{n+1} + 2 \sum_{j \in I} (\lambda_j - 1).$$

With this notation we may write

$$\begin{aligned} f(w, I) &:= \binom{w}{n} - \sum_{j \in I} \binom{w_j}{1} \\ &= s + \sum_{j \in I} (1 - \lambda_j) \binom{-w_j}{1} + \sum_{j \in I^c} \lambda_j \binom{w_j}{1} + \lambda_I e_{n+1} \end{aligned}$$

Hence, if the scalar λ_I is positive then the lattice point $f(w, I)$ is contained in some $s + Q_{k'}$, say, and therefore, it contributes to $a_{n-i}(P)$ (cf. (2.2)).

Since $s_{n+1} < 0$ we have $\sum_{j=1}^{n+1} \lambda_j > n$ (cf. (2.3)) and so we get either $\lambda_I > 0$ or $\lambda_{I^c} > 0$. In other words, either $f(w, I)$ contributes to $a_{n-i}(P)$ or $f(w, I^c)$ contributes to $a_i(P)$. Since this argument works for any subset I of cardinality i the lattice point $(w, n)^\top$ "produces" in this way a contribution of $\binom{n}{i}$ to the sum $a_i(P) + a_{n-i}(P)$.

Next we have to check that for two different points

$$\binom{w}{n} = s + \sum_{j=1}^n \lambda_j \binom{w_j}{1} + \lambda_{n+1} e_{n+1} \text{ and } \binom{\tilde{w}}{n} = s + \sum_{j=1}^n \mu_j \binom{\tilde{w}_j}{1} + \mu_{n+1} e_{n+1},$$

the lattice points $f(w, I)$ and $f(\tilde{w}, \tilde{I})$, $\#I = \#\tilde{I} = i$, are also different, provided both of them contribute to $a_{n-i}(P)$. Suppose the opposite, i.e., $f(w, I) = f(\tilde{w}, \tilde{I})$. Since both of them contribute to $a_{n-i}(P)$ the two points $f(w, I)$, $f(\tilde{w}, \tilde{I})$ lie in the same cone $s + C_{k'}$, say, and since any lattice point in $s + C$ is contained in exactly one of the simplicial cones $s + C_k$ we conclude

$$\{-w_j : j \in I\} \cup \{w_j : j \in I^c\} = \{-\tilde{w}_j : j \in \tilde{I}\} \cup \{\tilde{w}_j : j \in \tilde{I}^c\}.$$

If $\{-w_j : j \in I\} = \{-\tilde{w}_j : j \in \tilde{I}\}$ then we must also have $\{w_j : j \in I^c\} = \{\tilde{w}_j : j \in \tilde{I}^c\}$. Each point in a simplicial cone, however, has a unique representation with respect to the generators and so we get the contradiction $(w, n)^\top = (\tilde{w}, n)^\top$. Therefore, we may assume that there exists a $j_1 \in I \cap \tilde{I}^c$ and a $j_2 \in \tilde{I} \cap I^c$. Thus $1 - \lambda_{j_1} = \mu_{j_1}$ and $1 - \mu_{j_2} = \lambda_{j_2}$ and so

$$(2.4) \quad \mu_{j_1} + \mu_{j_2} + \lambda_{j_1} + \lambda_{j_2} = 2.$$

On the other hand, since $\sum_{i=1}^{n+1} \lambda_i, \sum_{i=1}^{n+1} \mu_i > n$ and $\lambda_i, \mu_i < 1$ we have $\lambda_{j_1} + \lambda_{j_2}, \mu_{j_1} + \mu_{j_2} > 1$ contradicting (2.4).

So far we have shown that

$$(2.5) \quad a_i(P) + a_{n-i}(P) \geq \binom{n}{i} a_n(P).$$

Now there is one special point $\binom{\hat{w}}{n}$ which contributes to $a_i(P)$ as well as to $a_{n-i}(P)$. Since the origin $0 \in \mathbb{R}^{n+1}$ is contained in one of the cones $s + C_k$, say,

we can find a representation of the form

$$0 = s + \sum_{j=1}^n \mu_j \begin{pmatrix} w_j \\ 1 \end{pmatrix} + \mu_{n+1} e_{n+1},$$

$\mu_i > 0$. Choosing the vector s sufficiently small we may assume that

$$(2.6) \quad \mu_{n+1} + 2 \sum_{j=1}^n \mu_j < 1.$$

Hence the vector

$$\begin{pmatrix} \widehat{w} \\ n \end{pmatrix} = \sum_{j=1}^n \begin{pmatrix} -w_j \\ 1 \end{pmatrix} = s + \sum_{j=1}^n (1 - \mu_j) \begin{pmatrix} -w_j \\ 1 \end{pmatrix} + \left(\mu_{n+1} + 2 \sum_{j=1}^n \mu_j \right) e_{n+1}$$

is contained in some $s + Q_{k'}$, say. On account of (2.6) the vectors $f(\widehat{w}, I)$ and $f(\widehat{w}, I^c)$ are contained in some of these parallelepipeds for all subsets $I \subset \{1, \dots, n\}$ of cardinality i . Thus the vector $(\widehat{w}, n)^\top$ gives a contribution of $\binom{n}{i}$ to $a_i(P)$ and to $a_{n-i}(P)$. Together with (2.5) this proves the lemma. \square

For the proof of the inequalities in Remark (1.6) we just observe that the last part of the proof above where the vector $(\widehat{w}, n)^\top$ is considered, in particular implies that $a_i(P) \geq \binom{n}{i}$ for any lattice crosspolytope. Now any n -dimensional 0-symmetric lattice polytope \tilde{P} contains a 0-symmetric lattice crosspolytope P and by Stanley's Monotonicity Theorem [37] (see also [5]) we have $a_i(\tilde{P}) \geq a_i(P)$.

Next we come to the roots of the polynomial $G(s, S_n(1))$ which on account of Proposition 1.3 is given by

$$(2.7) \quad G(s, S_n(1)) = \sum_{i=0}^n \binom{s+n-i}{n} = \binom{s+n+1}{n+1} - \binom{s}{n+1}.$$

Proof of Theorem 1.7. One way to see that all roots of $G(s, S_n(1))$ have real part $-1/2$ is to apply a theorem of Rodriguez-Villegas [34]. In our setting it says that if all roots of the polynomial $f(s, P) = \sum_{i=0}^n a_i(P) s^i$ lie on the unit circle then all roots of $G(s, P)$ have real part $-1/2$. In our case we have $f(s, S_n(1)) = \sum_{i=0}^n s^i$ and so the norm of each root of that polynomial is 1.

Now let $s_0 = -1/2 + i b = r_0 e^{i\alpha_0}$ be a point on the line with real part $-1/2$ where we assume $b \geq 0$. Furthermore, for $m = 1, \dots, n$ let $s_0 - m = r_m e^{i\alpha_m}$. Since $|s_0 - m| = |s_0 + m + 1|$, $m = 0, \dots, n$, we also know that $s_0 + m + 1 = r_m e^{i(\pi - \alpha_m)}$. From the right hand side of (2.7) we conclude that s_0 is a root of $G(s, S_n(1))$ if and only if

$$(s_0 + n + 1)(s_0 + n) \cdots (s_0 + 1) = s_0(s_0 - 1) \cdots (s_0 - n).$$

Substituting the polar representations leads to

$$(-1)^{n+1} = e^{i(2\alpha_0 + 2\alpha_1 + \dots + 2\alpha_n)}$$

Replacing the angle α_m by $\pi/2 + \overline{\alpha_m}$, $\overline{\alpha_m} \in (0, \pi/2]$, gives $1 = e^{i(2\overline{\alpha_0} + 2\overline{\alpha_1} + \dots + 2\overline{\alpha_n})}$ and thus we must have

$$\overline{\alpha_0} + \overline{\alpha_1} + \dots + \overline{\alpha_n} = k\pi.$$

for an integer $k \in \{1, \dots, \lfloor (n+1)/2 \rfloor\}$. Observe, that we have assumed $b \geq 0$. By construction we have $\cot \bar{\alpha}_m = 2b/(2m+1)$, $m = 0, \dots, n$, and so we get that $s_0 = -1/2 + ib$ is a root of $G(s, S_n(1))$ if and only if

$$h(b) := \sum_{m=0}^n \cot^{-1} \left(\frac{2b}{2m+1} \right) \in \{\pi, 2\pi, \dots, \lfloor (n+1)/2 \rfloor \pi\},$$

where we require $\cot^{-1}(\cdot) \in (0, \pi/2]$. Since $h(b)$ is a monotonously decreasing function in b the imaginary part b_n of the root of maximal norm is determined by the equation $h(b_n) = \pi$. Since $\cot^{-1}(t) = \tan^{-1}(1/t)$ “the inverse” of the cotangent has the power series representation $\cot^{-1}(t) = \sum_{k=0}^{\infty} (-1)^k / (2k+1) (1/t)^{2k+1}$ for $t > 1$. So we have $1/t > \cot^{-1}(t) > 1/t - 1/(3t^3)$. Hence for $b > n + 1/2$ we may write

$$\frac{1}{b} \frac{(n+1)^2}{2} > h(b) > \frac{1}{b} \frac{(n+1)^2}{2} - c \frac{n^4}{b^3}$$

for a suitable constant c . Thus $b_n = n(n+2)/(2\pi) + o(n)$. \square

3. REFLEXIVE POLYTOPES

As mentioned in the introduction reflexive polytopes are of particular interests in many different branches of mathematics and have a lot of nice geometric properties. Some of them are collected in the following lemma for which we refer to [1, 19].

Lemma 3.1. *Let $P \in \mathcal{P}^n$ with $0 \in \text{int}(P)$. Then P is reflexive if and only if*

- i) $P^\star \in \mathcal{P}^n$.
- ii) $a_i(P) = a_{n-i}(P)$, $0 \leq i \leq n$.
- iii) $G(kP) = G((k+1)\text{int}(P))$ for $k \in \mathbb{N}$.
- iv) $\text{vol}(P) = (n/2) G_{n-1}(P)$, i.e., the origin lies in an adjacent lattice hyperplane to any facet.

In particular, the origin is the only interior lattice point of a reflexive polytope, and reflexive polytopes are precisely those lattice polytopes satisfying the functional equation:

$$G(s, P) = (-1)^n G(-(1+s), P), \quad s \in \mathbb{C}.$$

Hence in any odd dimension the Ehrhart polynomials of reflexive polytopes have the real root $-1/2$.

Now let $P \in \mathcal{P}^n$ be a lattice polytope such that the real part of all roots $-\gamma_i(P)$ of its Ehrhart polynomial is $-1/2$. Then from (1.5) we immediately get

$$\frac{n}{2} G_{n-1}(P) = \text{vol}(P) \leq 2^n$$

which by Lemma 3.1 iv) verifies Proposition 1.8.

In dimension 2 any lattice polygon P whose only interior lattice point is the origin is reflexive and its Ehrhart polynomial is given by (cf. (1.3), (1.4))

$$G(s, P) = \text{vol}(P) \left(s^2 + s + \frac{1}{\text{vol}(P)} \right).$$

Thus all roots have real part $-1/2$ if and only if $\text{vol}(P) \leq 4$. Among the well known 16 reflexive polytopes in \mathbb{R}^2 (cf. e.g. [33]) there is only one with volume bigger than 4, namely the simplex $S = -(1, 1)^\top + \text{conv}\{0, 3e_1, 3e_2\}$ of volume $9/2$. By Theorem 1.2 we know that the reflexive polygon of minimal volume is $S_2(1)$ of volume $3/2$. Hence the Cartesian product $S \times S_2(1)$ is an example of a 4-dimensional reflexive polytope of volume less than 2^n ($n = 4$), but not all roots of its Ehrhart polynomial have real part $-1/2$.

Proof of Proposition 1.9. First we check that all roots of the Ehrhart polynomial of a 3-dimensional reflexive polytope P have real part $-1/2$ if and only if its volume is not bigger than 8. By Lemma 3.1 ii) we have $a_1(P) = a_2(P)$ and so (cf. (1.3), (1.4))

$$\begin{aligned} G(s, P) &= \frac{1}{6} [(2a_1(P) + 2)s^3 + (3a_1(P) + 3)s^2 + (13a_1(P) + 1)s + 6] \\ &= \text{vol}(P) \left[s^3 + \frac{3}{2}s^2 + \left(\frac{1}{2} + \frac{2}{\text{vol}(P)} \right) s + \frac{1}{\text{vol}(P)} \right] \\ &= \text{vol}(P) \left[\left(s + \frac{1}{2} \right) \left(s^2 + s + \frac{2}{\text{vol}(P)} \right) \right]. \end{aligned}$$

Hence all roots have real part $-1/2$ iff $\text{vol}(P) \leq 8$.

Now let P be a 4-dimensional reflexive polytope. Again by Lemma 3.1 we have $a_1(P) = a_3(P)$ and so we find

$$\begin{aligned} G(s, P) &= \frac{1}{24} [(2a_1(P) + a_2(P) + 2)s^4 + (4a_1(P) + 2a_2(P) + 4)s^3 \\ &\quad + (10a_1(P) - a_2(P) + 46)s^2 + (8a_1(P) - 2a_2(P) + 44)s + 24] \\ &= \text{vol}(P) \left[s^3 + 2s^3 + (2\mu + 1)s^2 + (2\mu)s + \frac{1}{\text{vol}(P)} \right], \end{aligned}$$

where $\mu = (1 + (1/4)a_1(P))/\text{vol}(P) - 1$. Further we set $\beta = 1/\text{vol}(P)$ and obtain

$$G(s, P) = \text{vol}(P) \left[\left(s^2 + s + \mu + \sqrt{\mu^2 - \beta} \right) \cdot \left(s^2 + s + \mu - \sqrt{\mu^2 - \beta} \right) \right].$$

Thus all roots have real part $-1/2$ if and only if $\mu^2 \geq \beta$ and $\mu - \sqrt{\mu^2 - \beta} \geq 1/4$. The first condition translates into $(2 + (1/2)a_1(P) - 2\text{vol}(P))^2 \geq 4\text{vol}(P)$ and the second becomes $2a_1(P) \leq 9\text{vol}(P) + 8$. Since $a_1(P) = G(P) - 5$ we get the inequalities stated in Proposition 1.9. \square

Thanks to the classification of Kreuzer and Skarke (cf. <http://hep.itp.tuwien.ac.at/~kreuzer/CY/>) one can check that among the 4319 reflexive polytopes in dimension 3 only 64 have volume bigger than 8 and that there are only 33 different Ehrhart polynomials corresponding to $a_1(P) \in \{1, \dots, 35\} \setminus \{33, 34\}$.

In dimension 4 we have just made some calculations with the 1561 reflexive simplices (cf. [10]). Here the Ehrhart polynomials of "only" 574 of them have roots with real part $-1/2$. Finally we present two 4-dimensional reflexive simplices which show that both conditions in Proposition 1.9 are necessary. The first simplex is given by the inequalities $E_1 = \{x \in \mathbb{R}^4 : x_i \geq -1, 1 \leq i \leq 3, -x_3 - 2x_4 \leq 2, x_1 + x_2 + 2x_3 + 2x_4 \leq 1\}$. With the help of the computer

program **latte** [11], which we have used for all our calculations, one (the computer) can easily determine the Ehrhart polynomial of such a polytope and here we find

$$G(s, E_1) = \frac{27}{2} s^4 + 27 s^3 + 21 y^2 + \frac{15}{2} y + 1.$$

Thus we have $G(E_1) = 70$ and hence $(G(E_1) - 1 - 4 \operatorname{vol}(E_1))^2 \geq 16 \operatorname{vol}(E_1)$ but $2G(E_1) > 9 \operatorname{vol}(E_1) + 18$. Next let $E_2 = \{x \in \mathbb{R}^4 : -x_1 \leq 1, -x_2 \leq 1, -2x_1 - 3x_2 - 4x_3 \leq 1, -4x_1 - 5x_2 - 8x_4 \leq 1, 10x_1 + 9x_2 + 4x_3 + 8x_4 \leq 1\}$. Then

$$G(s, E_2) = \frac{4}{3} s^4 + \frac{8}{3} s^3 + \frac{8}{3} y^2 + \frac{4}{3} y + 1.$$

In this case we have $G(E_2) = 9$, $2G(E_2) \leq 9 \operatorname{vol}(E_2) + 18$ but $(G(E_2) - 1 - 4 \operatorname{vol}(E_2))^2 < 16 \operatorname{vol}(E_2)$.

4. 3-DIMENSIONAL LATTICE POLYTOPES

In this section we will study the roots of Ehrhart polynomials of 3-dimensional lattice polytopes. To this end we will distinguish polytopes with and without interior lattice points.

Theorem 4.1. *Let $\Gamma(3, 0)$ be the set of all roots of Ehrhart polynomials of 3-dimensional lattice polytopes $P \in \mathcal{P}^3(0)$, i.e., without interior lattice points.*

- i) $\Gamma(3, 0) \cap \mathbb{R} = \{-3, -2\} \cup (-2, 1)$. Moreover, 1 is a cluster point and there are infinitely many roots in the interval $(-2, -1)$.
- ii) $\{a + ib \in \Gamma(3, 0) : b \neq 0\} \subset W := \{a + ib : (a+1)^2 + b^2 \leq 2 \text{ and } a \geq -1\}$.
- iii) On the boundary of the semicircle W lie exactly 33 pairs of zeros. $-1 \pm i\sqrt{2}$, $-1 \pm i/\sqrt{2}$, $-1 \pm i$ and $-1 \pm i/\sqrt{5}$ are the only complex roots in $\Gamma(3, 0)$ with real part -1 .

For the proof we need the following proposition

Proposition 4.2. *Let $P \in \mathcal{P}_n$ and let $k \in \mathbb{N}$ be the smallest positive integer with $G(k \operatorname{int} P) \neq 0$. Then*

$$G_{n-1}(P) \leq \frac{nk}{2} G_n(P) = \frac{nk}{2} \operatorname{vol}(P).$$

Proof. Let $P = \{x \in \mathbb{R}^n : a_j x \leq b_j, 1 \leq j \leq m\}$ be a lattice polytope with facets F_j corresponding the outer normal vector a_j . It was already shown by Ehrhart [14] that

$$(4.1) \quad G_{n-1}(P) = \frac{1}{2} \sum_{i=1}^m \frac{\operatorname{vol}_{n-1}(F_i)}{\det(\operatorname{aff} F_i \cap \mathbb{Z}^n)},$$

where $\operatorname{vol}_{n-1}()$ denotes the $(n-1)$ -dimensional volume and $\det(\operatorname{aff} F_i \cap \mathbb{Z}^n)$ denotes the determinant of the $(n-1)$ -dimensional sublattice of \mathbb{Z}^n contained in the affine hull of the facet F_i .

Since P is a lattice polytope we can assume $a_j \in \mathbb{Z}^n$, $0 \in P$, $b_j \in \mathbb{N}$, and that the vectors a_j are primitive, i.e., $\operatorname{conv}\{0, a_j\} \cap \mathbb{Z}^n = \{0, a_j\}$. Hence

$\det(\text{aff } F_j \cap \mathbb{Z}^n) = \|a_j\|$, where $\|\cdot\|$ denotes the Euclidean norm. By the choice of k we can find a $z \in \mathbb{Z}$ such that $(1/k)z \in \text{int } P$ and so we find (cf. (4.1))

$$\begin{aligned} \text{vol}(P) &= \frac{1}{n} \sum_{i=1}^m \text{vol}_{n-1}(F_i) \frac{|a_j(1/k)z - b_j|}{\|a_j\|} \geq \frac{2}{nk} \frac{1}{2} \sum_{i=1}^m \frac{\text{vol}_{n-1}(F_i)}{\|a_j\|} \\ &= \frac{2}{nk} G_{n-1}(P) \end{aligned}$$

□

We remark that we always have $k \leq n+1$ and thus by Proposition 4.2 $G_{n-1}(P) \leq \binom{n+1}{2} \text{vol}(P)$ which is a special case of another series of inequalities proved in [7]. The case $k=1$ and thus $G_{n-1}(P) \leq (n/2) \text{vol}(P)$ was already shown in [38]. So with the notation of Proposition 4.2 we have for three-dimensional polytopes P

$$(4.2) \quad 1 \leq G_2(P) \leq k \frac{3}{2} \text{vol}(P),$$

where the lower bound follows from (4.1) and the fact that for any facet $\text{vol}_{n-1}(F_i)/\det(\text{aff } F_i \cap \mathbb{Z}^n) \geq 1/(n-1)!$.

Proof of Theorem 4.1. From [2, Theorem 1.2, Proposition 4.7] it follows that all real roots of Ehrhart polynomials of 3-dimensional polytopes are within $[-3, 1)$ and in [17, Theorem 1.7] it was shown that 1 is cluster point of $\Gamma(3, 0)$. Next we observe that -1 is a root of $G(s, P)$ for any polytope without interior lattice points (cf. (1.2)). Hence, denoting for short the coefficients $G_i(P)$ by g_i we have $g_3 - g_2 + g_1 - 1 = 0$ and so may write

$$G(s, P) = g_3 s^3 + g_2 s^2 + g_1 s + 1 = g_3 (s+1) \left(s^2 + \frac{g_2 - g_3}{g_3} s + \frac{1}{g_3} \right).$$

For the two remaining roots $-\gamma_{1,2}$ we find

$$(4.3) \quad -\gamma_{1,2} = -\frac{g_2 - g_3}{2g_3} \pm \sqrt{\left(\frac{g_2 - g_3}{2g_3} \right)^2 - \frac{1}{g_3}}.$$

Now we want to show that there are no real roots in $(-3, -2)$. Suppose -2 is another root of $G(s, P)$, then for the third root γ , say, we get $\gamma = -1/(2g_3)$. Since $6g_3$ is an integer we conclude that $\gamma = -3$ or $\gamma \geq -3/2$. Hence if there is an Ehrhart-polynomial having a real root in $(-3, -2)$ then we know $G(2 \text{int } P) \neq 0$ and so by (4.2) $g_2 \leq 3g_3$. For given g_3 the right hand side in (4.3) becomes minimal if g_2 is as large as possible. Thus $-\gamma_{1,2} \geq -1 \pm \sqrt{1 - 1/g_3} > -2$. Observe that $g_3 \geq 1$ since we have assumed that all roots are real and $g_2 \leq 3g_3$.

For i) it remains to show that there are infinitely many real roots in $(-2, -1)$. To this end we consider for an integer q the pyramids $P(q) = \text{conv}\{0, 2e_1, qe_2, 2e_1 + qe_2, e_3\}$. Then one gets $G_3(P(q)) = 2/3 q$ and $G_2(P(q)) = 3/2 q$ which shows by (4.3) that for q large $G(s, P(q))$ has a real root in $(-2, -1)$ depending on q .

For ii) we assume that the roots $-\gamma_{1,2}$ in (4.3) are complex. Writing $-\gamma_{1,2} = a \pm ib$ leads to $b^2 = 1/g_3 - a^2$. Since $1/g_3 = (1 - 2a)/g_2$ we may rewrite this as

$$\left(a + \frac{1}{g_2}\right)^2 + b^2 = \left(\frac{1}{g_2}\right)^2 + \frac{1}{g_2}.$$

By (4.2) we know $g_2 \geq 1$ and it is not hard to see that all the circles above are contained in the disk given by the largest one, i.e., we have $(a + 1)^2 + b^2 \leq 2$. Since we assume that the roots $-\gamma_{1,2}$ are complex we have $G(2 \operatorname{int} P) \neq 0$, because otherwise -2 would be a root. Thus from (4.2) we conclude $g_2 \leq 3g_3$ which is equivalent to $a = -(g_2 - g_3)/(2g_3) \geq -1$.

Now we come to part iii). Let $g_3 = \operatorname{vol}(P) = k/6$, $k \in \mathbb{N}$. All complex roots on the semicircle satisfy $g_2 = 1$ and

$$0 > \left(\frac{g_2 - g_3}{2g_3}\right)^2 - \frac{1}{g_3} = \left(\frac{3}{k} - \frac{1}{2}\right)^2 - \frac{6}{k} = \frac{1}{4k^2} (k^2 - 36k + 36).$$

Hence k is restricted to the integers $k = 2, \dots, 34$. The Reeve-simplices $T(k) = \operatorname{conv}\{0, e_1, e_2, (1, 1, k)^\top\}$ form a family of simplices whose Ehrhart polynomials

$$G(S, T(k)) = \frac{k}{6} s^3 + s^2 + \frac{12 - k}{6} s + 1$$

have these roots.

Finally we consider the case that the complex roots have real part -1 . Then $g_2 = 3g_3$ and the Ehrhart polynomial of such a polytope P is of the type

$$G(s, P) = g_3 s^3 + 3g_3 s^2 + (2g_3 + 1)s + 1.$$

The roots of that polynomial are given by $-1, -1 \pm \sqrt{1 - 1/g_3}$. Again let $g_3 = k/6$, $k \in \mathbb{N}$. Since $1 - 1/g_3$ has to be negative and since $g_3 = g_2/3 \geq 1/3$ we just have to consider the cases $k = 2, \dots, 5$. Moreover we note that for such a polytope P all roots of $2P$ have real part $-1/2$ and so $2P$ has to be a reflexive polytope (cf. Proposition 1.8). Hence all possible candidates are contained in database of Kreuzer and Skarke of 3-dimensional reflexive polytopes.

An example for $k = 2$ is given by the Reeve-simplex $T(2)$ with Ehrhart polynomial $(1/3)s^3 + s^2 + 5/3s + 1$ and with complex roots $-1 \pm i\sqrt{2}$. For $k = 3, 4$ we found respectively the simplices $\operatorname{conv}\{0, e_1, e_2, (2, 2, 3)^\top\}$ with complex roots $-1 \pm i$ and for $k = 4$ the simplex $\operatorname{conv}\{0, e_1, e_2, (2, 3, 4)^\top\}$ and complex roots $-1 \pm 1/\sqrt{2}i$. For $k = 5$, i.e., $g_3 = 5/6$, there does not exist a simplex with the required Ehrhart polynomial. However, the pyramid over a quadrangle given by $\operatorname{conv}\{0, e_1, 2e_2, 2e_1 + e_2, e_3\}$ has the Ehrhart polynomial $5/6s^3 + 5/2s^2 + 16/6s + 1$ with complex roots $-1 \pm i/\sqrt{5}$. □

Next we come to 3-dimensional polytopes with interior lattice points. For those lattice polytopes we know by Proposition 4.2 that

$$(4.4) \quad G_2(P) \leq \frac{3}{2} G_3(P).$$

First we state some simple properties on the real parts of the roots.

Proposition 4.3. *Let $P \in \mathcal{P}_n(l)$, $l \geq 1$.*

- i) If all roots of $G(s, P)$ are real then either all roots are contained in $(-1, 0)$ or one belongs to $(-1, 0)$ and the two others are in $(0, 1)$.
- ii) If $G(s, P)$ has only one real root γ then $\gamma \in (-1, 0)$ and the real parts of the complex roots are contained in $(-3/4, 1/2)$.

Proof. Let us assume that all roots are real. The point of inflexion of the real polynomial $G(t, P)$, $t \in \mathbb{R}$, is given by $-G_2(P)/(3G_3(P))$ which by (4.4) is contained in $[-1/2, 0)$. Furthermore the derivative of that polynomial at 0 is given by $G_1(P)$ and we also know that $G(-1, P) = -l < 0$, $G(1, P) > 0$. Thus, the polynomial has always a real root in $(-1, 0)$. If all roots are real then two cases occur. If $G_1(P) \geq 0$ then all of them are in $(-1, 0)$ and otherwise one root is contained in $(-1, 0)$ and the positive roots are strictly less than 1.

Now suppose we have one real root γ and the two complex roots $a \pm ib$. Since $(1/3)(2a + \gamma) = -G_2(P)/(3G_3(P)) \in [-1/2, 0)$ and $\gamma \in (-1, 0)$ we must have $-3/4 < a < 1/2$. \square

For the proof of Theorem 1.7 we also need the following lemma.

Lemma 4.4. *Let $P \in \mathcal{P}_n(l)$, $l \geq 1$. Then*

- i) $G_1(P) \leq G_2(P) + G_3(P) + \frac{2}{3} \leq \frac{5}{2}G_3(P) + \frac{2}{3}$,
- ii) $G(-1/(3 \operatorname{vol}(P)), P) \geq 0$,

and both bounds are tight. In particular, equality in ii) is only attained if P is unimodular equivalent to $S_3(1)$.

Proof. By (1.2) we have $G_1(P) = l - G_3(P) + G_2(P) + 1$. By Theorem 1.2 we also know $l \leq 2G_3(P)/2 - 1/3$ and thus $G_1(P) \leq G_2(P) + G_3(P) + \frac{2}{3}$. The second inequality in i) is just a consequence of (4.4).

For the proof of ii) we write for short g_i instead of $G_i(P)$. On account of i) we get

$$\begin{aligned} G\left(-\frac{1}{3 \operatorname{vol}(P)}, P\right) &= -\frac{1}{27(g_3)^2} + \frac{g_2}{9(g_3)^2} - \frac{g_1}{3g_3} + 1 \\ &\geq -\frac{1}{27(g_3)^2} + \frac{g_2}{9(g_3)^2} - \frac{g_2 + g_3 + 2/3}{3g_3} + 1 \\ &= \frac{1}{27(g_3)^2} (3g_2 - 1 - 6g_3) - \frac{g_2}{3g_3} + \frac{2}{3}. \end{aligned}$$

With $g_2 \geq 1$ (cf. (4.2)) and $g_2/(3g_3) \leq 1/2$ (cf. (4.4)) we obtain

$$G\left(-\frac{1}{3 \operatorname{vol}(P)}, P\right) \geq \frac{1}{27(g_3)^2} (2 - 6g_3) + \frac{1}{6} = \frac{2}{3} \left(\frac{1}{3g_3} - \frac{1}{2} \right)^2 \geq 0.$$

Now as the inequalities show we have equality in i) if and only if $\operatorname{vol}(P) = (3l+1)6$, i.e., if we have equality in Theorem 1.2. In ii) we have equality if and only if in addition $\operatorname{vol}(P) = 2/3$, i.e., $l = 1$ and so $P = S_3(1)$. \square

Proof of Theorem 1.7. In view of Theorem 4.1 and Proposition 4.3 it remains to show that the norm of each complex root of the Ehrhart polynomial of a polytope with interior lattice points is bounded by $\sqrt{3}$. Let $-\gamma_1$ be the real root and $a \pm ib$ be the complex roots with $b \neq 0$. Since $G(-1, p) = -l < 0$ we

get from Lemma 4.4 i) that $\gamma_1 \geq 1/(3\text{vol}(P))$. On the other hand we know that $\gamma_1 \cdot (a^2 + b^2) = 1/\text{vol}(P)$ (cf. (1.5)) and thus $(a^2 + b^2) \leq 3$.

Among the polytopes $P \in \mathcal{P}_3(l)$, $l \leq 1$, the bound on the norm is attained if and only if the polynomial has two complex roots $a \pm ib$ and one real root $-\gamma_1$ (cf. Proposition 4.3). Since $\gamma_1 \cdot (a^2 + b^2) = 1/\text{vol}(P)$ we get $-\gamma_1 = -1/(3\text{vol}(P))$. Thus by Lemma 4.4 ii) we conclude that this is only the case for a polytope unimodular equivalent to $S_3(1)$. By Proposition (1.3) we have $G(s, S_3(1)) = 2/3 s^3 + s^2 + 7/3 s + 1$ with roots $-1/2, -1/2 \pm i\sqrt{11}/2$. \square

REFERENCES

- [1] V.V. Batyrev, *Dual polyhedra and mirror symmetry for calabi-yau hypersurfaces in toric varieties*, J. Algebr. Geom. **3** (1994), 493–535.
- [2] M. Beck, J. De Loera, M. Develin, J. Pfeifle, and R.P. Stanley, *Coefficients and roots of Ehrhart polynomials*, Contemp. Math. **374** (2005), 15–36.
- [3] M. Beck and D. Pixton, *The Ehrhart polynomial of the Birkhoff polytope*, Discrete Comput. Geom. **30** (2003), no. 4, 623–637.
- [4] M. Beck and S. Robins, *Computing the continuous discretely: Integer-point enumeration in polyhedra*, Springer, (to appear), Preprint at <http://math.sfsu.edu/beck/papers/ccd.html>.
- [5] M. Beck and F. Sottile, *Irrational proofs for three theorems of Stanley*, to appear in European J. Combinatorics, preprint at <http://front.math.ucdavis.edu/math.CO/0501359>.
- [6] U. Betke and P. Gritzmann, *An application of valuation theory to two problems of discrete geometry*, Discrete Math. **58** (1986), 81–85.
- [7] U. Betke and P. McMullen, *Lattice points in lattice polytopes*, Monatsh. Math. **99** (1985), no. 4, 253–265.
- [8] B. Braun, *Norm bounds for Ehrhart polynomials*, preprint at <http://arxiv.org/abs/math.CO/0602464>.
- [9] D. Bump, K.-K. Choi, P. Kurlberg, and J. Vaaler, *A local Riemann hypothesis I*, Math. Z. **233** (2000), no. 1, 1–19.
- [10] H. Conrads, *Weighted projective spaces and reflexive simplices*, Manuscr. Math. **107** (2002), no. 2, 215–227.
- [11] J. De Loera, D. Haws, R. Hemmecke, and P. Huggins, *A user's guide for latte v1.1, software package latte*, 2004, available at <http://www.math.ucdavis.edu/~latte>.
- [12] R. Diaz and S. Robins, *The Ehrhart polynomial of a lattice polytope*, Ann. of Math. **145** (1997), no. 3, 503–518, Erratum in 146:1 (1997), 237.
- [13] E. Ehrhart, *Sur les polyèdres rationnels homothétiques à n dimensions*, C. R. Acad. Sci., Paris, Sér. A **254** (1962), 616–618.
- [14] ———, *Sur un problème de géométrie diophantienne linéaire*, J. Reine Angew. Math. **227** (1967), 25–49.
- [15] ———, *Sur la loi de réciprocité des polyèdres rationnels*, C. R. Acad. Sci., Paris, Sér. A **266** (1968), 695–697.
- [16] P. M. Gruber and C. G. Lekkerkerker, *Geometry of numbers*, second ed., vol. 37, North-Holland Publishing Co., Amsterdam, 1987.
- [17] M. Henk, A. Schürmann, and J.M. Wills, *Ehrhart polynomials and successive minima*, to appear in Mathematika, preprint at <http://front.math.ucdavis.edu/math.MG/0507528>.
- [18] D. Hensley, *Lattice vertex polytopes with interior lattice points*, Pac. J. Math. **105** (1983), 183–191.
- [19] T. Hibi, *Dual polytopes of rational convex polytopes*, Combinatorica **12** (1992), no. 2, 237–240.

- [20] ———, *A lower bound theorem for Ehrhart polynomials of convex polytopes*, Adv. Math. **105** (1994), no. 2, 162–165.
- [21] P. Kirschenhofer, A. Pethö, and R.T. Tichy, *On analytical and diophantine properties of a family of counting polynomials*, Acta Sci. Math. **65** (1999), 47–59.
- [22] M. Kreuzer and H. Skarke, *Classification of reflexive polyhedra in three dimensions*, Adv. Theor. Math. Phys. **2** (1998), 853–871.
- [23] ———, *Classification of reflexive polyhedra in four dimensions*, Adv. Theor. Math. Phys. **4** (2000), 1209–1230.
- [24] J.C. Lagarias and G.M. Ziegler, *Bounds for lattice polytopes containing a fixed number of interior points in a sublattice*, Canad. J. Math. **43** (1991), 1022–1035.
- [25] F. Liu, *Ehrhart polynomials of lattice-face polytopes*, <http://arxiv.org/abs/math.CO/0512616>.
- [26] ———, *Ehrhart polynomials of cyclic polytopes*, J. Comb. Theory, Ser. A **111** (2005), 111–127.
- [27] L.J. Mordell, *Lattice points in tetrahedron and generalized Dedekind sums*, J. Indian Math. Soc. (N.S.) **15** (1951), 41–46.
- [28] M. Mustata and S. Payne, *Ehrhart polynomials and stringy Betti numbers*, <http://arxiv.org/abs/math.AG/0505054>.
- [29] B. Nill, *Gorenstein toric Fano varieties*, Ph.D. thesis, University Tübingen, 2005, <http://w210.ub.uni-tuebingen.de/dbt/volltexte/2005/1888/pdf/nill.pdf>.
- [30] G.A. Pick, *Geometrisches zur Zahlenlehre*, Sitzungsber. Lotus Prag **19** (1899), 311–319.
- [31] O. Pikhurko, *Lattice points in lattice polytopes*, Mathematika **48** (2001), no. 1-2, 15–24.
- [32] J.E. Pommersheim, *Toric varieties, lattice points and Dedekind sums*, Math. Ann. **295** (1993), no. 1, 1–24.
- [33] B. Poonen and F. Rodriguez-Villegas, *Lattice polygons and the number 12*, Am. Math. Mon. **107** (2000), no. 3, 238–250.
- [34] F. Rodriguez-Villegas, *On the zeros of certain polynomials*, Proc. Amer. Math. Soc. **130** (2002), 2251–2254.
- [35] R.P. Stanley, *Decompositions of rational convex polytopes*, Ann. Discrete Math. **6** (1980), 333–342.
- [36] ———, *On the number of faces of centrally-symmetric simplicial polytopes*, Graphs and Combinatorics **3** (1987), 55–66.
- [37] ———, *A monotonicity property of h -vectors and h^* -vectors*, European J. Combinatorics **14** (1993), no. 3, 251–258.
- [38] J.M. Wills, *On an analog to Minkowski's lattice point theorem*, The geometric vein, Springer, New York, 1981, pp. 285–288.
- [39] ———, *Minkowski's successive minima and the zeros of a convexity-function*, Monatsh. Math. **109** (1990), no. 2, 157–164.
- [40] J. Zaks, M. A. Perles, and J.M. Wills, *On lattice polytopes having interior lattice points*, Elem. Math. **37** (1982), no. 2, 44–46.

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